SECOND PUBLIC EXAMINATION
Honour School of Mathematics and Statistics Part A: Paper A12

A12 Simulation and Statistical Programming

TRINITY TERM: 1.5 hours

• Answers to the best two questions will count towards the total mark for the paper.
• All questions are worth 25 marks.
• You may hand in attempts to any number of questions.
• Begin the answer to each question in a new answer booklet.
• Hand in your answers in numerical order.
• Indicate on the front sheet the numbers of the questions attempted.
• A booklet with the front cover sheet completed must be handed in even if no question has been attempted.
• Cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such answer booklet and attach these answer booklets at the back of your work.

Do not turn this page until you are told that you may do so
1. Consider the probability density \( p^{(\alpha)}(x) = p^{(\alpha)}_{u}(x)/Z_\alpha \) where

\[
p^{(\alpha)}_{u}(x) = \begin{cases} x^\alpha, & 0 \leq x \leq 2, \\ 0, & \text{otherwise,} \end{cases}
\]

and where \( \alpha > 0 \) is a parameter and \( Z_\alpha = \int_{-\infty}^{\infty} p^{(\alpha)}_{u}(x)dx \) is a normalising constant.

(a) [5 marks] Show how to simulate \( X \sim p^{(\alpha)} \) by the method of inversion. Give the inverse cumulative distribution function explicitly.

(b) [15 marks]
   (i) Write down a rejection sampling algorithm which produces a sample \( X \sim p^{(\alpha)} \) using \( U(0, 2) \) random variables as proposals.
   (ii) Write an R function implementing the algorithm you wrote down in part (i). Your function should take as input the value of \( \alpha \) and an integer \( n > 0 \) and return as output a vector \( X \) containing \( n \) samples distributed according to \( p^{(\alpha)}(x) \).
   (iii) Calculate the expected number of trials per sample in the rejection sampling algorithm you wrote down in part (i).

(c) [5 marks] Consider the following algorithm:
   (1) Draw \( U, Y \sim U(0, 1) \).
   (2) If \( U \leq 2Y \), let \( X = Y \); otherwise start again from (1).

Derive the distribution of \( X \).
2. Suppose $p$ and $q$ are probability mass functions on $X = \{1, \ldots, m\}$ such that $q(x) > 0$ whenever $p(x) > 0$, and suppose $f : X \to \mathbb{R}$ is a given function.

(a) [6 marks] Give the importance sampling estimator for $E_p[f(X)]$ using $q$ as the proposal distribution, and show that this estimator is unbiased.

(b) [12 marks] In this part of the question we consider the proposal distribution $\hat{q}(x) = \frac{1}{m}$ for all $x \in X$ and the target distribution $\hat{p}(x) = c \exp(-\sqrt{x})$, $x \in X$ where $c$ is a normalising constant.

(i) Give a Metropolis-Hastings algorithm with target distribution $\hat{p}(x)$ using the proposal distribution $q(y \mid x) = \hat{q}(y)$.

(ii) Write an R function implementing the algorithm you wrote down in part (i). Your function should take as input integers $m > 0$ and $n > 0$ and it should return a vector of $n$ simulated values.

(c) [7 marks] Suppose that $h(z \mid x)$, $z \in \mathbb{N} = \{0, 1, \ldots\}$, is a probability mass function for each $x \in X$, such that

$$\sum_{z \in \mathbb{N}} zh(z \mid x) = p(x) \quad \text{for each } x \in X.$$

Consider a Markov chain $(X_k, Z_k)$ defined as follows: Start from $X_0 = 0$, $Z_0 = 1$ and for $k = 1, 2, \ldots$

- Draw $Y_k \sim q(\cdot \mid X_{k-1})$ and then $Z_k^* \sim h(\cdot \mid Y_k)$.
- Draw $U_k \sim U(0, 1)$ and set

$$(X_k, Z_k) = \begin{cases} (Y_k, Z_k^*), & \text{if } U_k \leq \frac{Z_k^*}{Z_{k-1}} \frac{q(X_{k-1} \mid Y_k)}{q(Y_k \mid X_{k-1})}, \\ (X_{k-1}, Z_{k-1}), & \text{otherwise.} \end{cases}$$

(i) Write down the transition probability

$$K((x, z), (x', z')) = P((X_n, Z_n) = (x', z') \mid (X_{n-1}, Z_{n-1}) = (x, z))$$

of this Markov chain, and show that the Markov chain is reversible with respect to the joint probability mass function $\pi(x, z) = h(z \mid x)z$, for $x \in X$ and $z \in \mathbb{N}$.

(ii) Suppose you know that this Markov chain is irreducible and has invariant probability $\pi$ given above. What can you say about

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(X_k)$$?

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3. Let $X = [X_1, \ldots, X_p]$ be an $n \times p$ real matrix of rank $p$, where $p < n$ and where $X_i$, $i = 1, \ldots, p$, are column vectors.

(a) [7 marks] Let $\beta = (\beta_1, \beta_2, \ldots, \beta_p)^T$ and $y = (y_1, y_2, \ldots, y_n)^T$ be real vectors. The normal equations are $X^T X \beta = X^T y$. Consider solving the normal equations for $\beta$.

(i) How many multiplications are needed to compute $X^T (X \beta)$? How many are needed to compute $(X^T X) \beta$? Which does R use to evaluate $t(X)^*%*%=beta$?

(ii) The QR factorization $X = QR$ of $X$ gives $X$ as the product of an $n \times p$ orthogonal matrix $Q$ (so that $Q^T Q = I$) and a $p \times p$ upper triangular matrix $R$ with positive entries on the diagonal. Show that if $R \beta = Q^T y$ then $\beta$ is a solution of the normal equations.

(b) [6 marks]

Let $Q = [Q_1, \ldots, Q_p]$ have orthonormal columns $Q_i$, $i = 1, \ldots, p$, and let $Q' = [Q_2, \ldots, Q_p]$. Let $R'$ be the $(p - 1) \times (p - 1)$ upper triangular matrix obtained from $R$ by deleting its first row and first column, let $r$ be the $1 \times (p - 1)$ row vector with entries $r = (R_{1,2}, R_{1,3}, \ldots, R_{1,p})$, and let $0_{(p-1)\times1}$ be a $(p - 1) \times 1$ column vector of zeros. Write $R$ in block form

$$R = \begin{pmatrix} R_{1,1} & r \\ 0_{(p-1)\times1} & R' \end{pmatrix}.$$ 

(i) Write down the relation $X = QR$ in terms of these objects and show that

$$R_{1,1} = |X_1|$$

$$Q_1 = X_1/R_{1,1}$$

$$r = Q_1^T [X_2, \ldots, X_p].$$

(ii) Show that if $X' = [X_2, \ldots, X_p] - Q_1 r$

and $Q'$ and $R'$ satisfy

$$X' = Q' R'$$

then $Q$ and $R$ satisfy $X = QR$.

(c) [4 marks] Give a recursive algorithm computing the QR factorisation of $X$.

(d) [8 marks] Write an R function implementing the algorithm you gave in (c). Your function should take as input a matrix $X$ and return as output the $Q$ and $R$ matrices in a list.