Answer all of Section A and two questions from Section B.

Start the answer to each question on a fresh page.

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight which the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.
Section A

1. Define the convolution of two functions \( f(x) \) and \( g(x) \) and use the definition to find an expression for the aperture transmission function of a double-slit arrangement, of slit widths \( b_1 \) and \( b_2 \) and separation \( d \).

Two functions \( f(x) \) and \( g(x) \) have Fourier transforms \( F(k) \) and \( G(k) \) respectively. Show that the Fourier transform of \( h(x) = f(x) \ast g(x) \) (the convolution of \( f(x) \) and \( g(x) \)), is given by
\[
H(k) = F(k) \times G(k).
\]

Use the convolution theorem and the aperture transmission function you found above, to write an expression for the Fraunhofer intensity pattern of the double slit arrangement. [Detailed derivation of the final result is not required]. [10]

2. Sketch the setup of a Michelson interferometer and explain the occurrence of localised circular fringes. Use your sketch to show how these fringes can be displayed on a screen and find an expression for the radii of the fringes of order ”m” in terms of the mirrors separation \( d \) and the focal length of the focusing lens \( f \). In a Michelson interferometer, fringes are observed due to monochromatic light. One mirror is moved 0.1029 mm and a shift of 400 fringes is observed. What is the wavelength of the light? A flake of glass of refractive index 1.686 is placed in one arm of the interferometer normal to the light beam. What must the thickness of glass be to change the pattern by 400 fringes? [10]

3. A Fabry-Perot etalon is formed by polishing and silvering the parallel surfaces of a plate of quarts of thickness \( t \) and refractive index \( n \). The etalon is illuminated by monochromatic, coherent light of frequency \( \nu \). Show that the transmitted intensity of the etalon may be written as
\[
T(\delta) \propto \frac{1}{1 + A \sin^2 \left( \frac{\delta}{2} \right)}.
\]

You may assume that the phase shift is \( \delta = (4\pi nt\nu/c) \cos \theta \), where \( \theta \) is the angle internal rays make to the surface normal. Give an expression for the parameter \( A \) in terms of the reflectivity \( R \) of the silvered faces of the etalon.

Sketch the transmission of the etalon as a function of the phase shift \( \delta \) for the case \( A \approx 10 \). Show that the full width at half maximum of the peaks in the transmission is given by
\[
\Delta \delta_{1/2} \approx \frac{4}{\sqrt{A}}.
\] [10]
4. Define the concepts: free spectral range, instrument bandwidth and resolving power of an optical spectrometer. Light emitted by a doublet of wavelength separation $\Delta \lambda$ and mean wavelength $\lambda = 500 \text{ nm}$ is incident normally on a transmission diffraction grating having $N = 1000$ grooves per millimetre and a width $W = 5 \text{ cm}$. What is the smallest value of $\Delta \lambda$ that can be resolved by the grating? Assume now that the light is incident on a Fabry-Perot etalon of refractive index $n = 1.5$, thickness $t = 1 \text{ mm}$ and faces reflection coefficient $R = 0.9$. Estimate the smallest $\Delta \lambda$ that the etalon can resolve. Explain how the resolving power of the grating can be improved by using a ”blazed” reflection grating. [You may use the resolving power formulas without derivation]. [10]
Section B

5. What is meant by the *Huygens-Fresnel* principle? A single slit of width $b$ is normally illuminated by a collimated beam of monochromatic light of wavelength $\lambda$. The intensity of the diffracted light when observed on a screen at a large distance from the slits is given by

$$I(\theta) = I_0 \left( \frac{\sin \beta}{\beta} \right)^2,$$

where $\theta$ is the angle of observation to the optical axis, $\beta = \frac{1}{2} kb \sin \theta$, $k = \frac{2\pi}{\lambda}$ and $I_0$ is a constant. Calculate the ratio of the intensity of the first maxima to the central maximum to two significant figures. [The first maxima for $\text{sinc}^2(x)$ occurs at $x = \pm 1.43\pi$.]

Two slits of width $b$ and separation $a$ ($a > b$) are normally illuminated by a collimated beam of monochromatic light of wavelength $\lambda$. Show that the intensity of the diffracted light when observed on a screen at a large distance from the slits is given by

$$I(\theta) = I_0 \left( \frac{\sin \beta}{\beta} \right)^2 \cos^2 \alpha,$$

where $\theta$ is the angle of observation to the optical axis, $\beta = \frac{1}{2} kb \sin \theta$, $\alpha = \frac{1}{2} ka \sin \theta$ and $k = \frac{2\pi}{\lambda}$. Given that the ratio of slit separation to slit width $r$ is chosen so that $2r + 1$ is an even integer, express the number of bright fringes seen under the central diffraction peak in the double-slit pattern in terms of $r$.

An aperture lies in the plane $z = 0$ and has an amplitude transmission function $T(y)$ (independent of $x$), given by

$$T(y) = \begin{cases} 1 - |y| & |y| < 1, \\ 0 & |y| \geq 1. \end{cases}$$

The aperture is illuminated by coherent, monochromatic light of wavelength $\lambda$ at normal incidence. Find an expression for the intensity of the diffraction pattern on a screen at a large distance from the aperture. What is the ratio of the intensity amplitude of the first maxima to the central maximum? Comment on this ratio compared to the ratio you obtained for a single slit. How is this relevant to the measurement of spectra?
6. A diffraction grating consists of $N$ identical long apertures of width $a$ separated by a distance $d$.

Obtain an expression for the angular distribution of light intensity diffracted from the grating. (Assume that $a$ is much smaller than the wavelength of light, so that the single aperture term is a constant, independent of angle.)

By considering that the principal maxima at wavelength $\lambda + \Delta\lambda$ occur at the same angle as the first minima for wavelength $\lambda$ in order for the two wavelengths to be resolved, obtain an expression for the theoretical resolving power of the grating.

Show that the range of wavelengths transmitted by a grating spectrometer (the bandpass) depends on the angular dispersion of the grating, the focal length of the collimating optics, and the width of the exit slit. Estimate the bandpass of a spectrometer which has a 100 mm wide grating with $d = (1/1800)$ mm. The spectrometer operates in first order, has 1 m focal length collimating optics, and a slit width of 100 $\mu$m. What value of slit width would yield a bandpass corresponding to the theoretical resolving power for $\lambda \approx 500$ nm?

7. Sketch the setup of a Michelson interferometer and explain how it is used as a Fourier-transform spectrometer.

Consider a monochromatic light source of wavenumber $\bar{\nu}$. Show that the intensity $I$ measured along the optical axis of the system as a function of the effective mirror separation $x$ is of the form

$$I(x) = \frac{1}{2}I(0)[1 + \cos(4\pi\bar{\nu}x)].$$

Assume now that a spectral line is split into three narrow components with wavenumbers $\bar{\nu}$ and $\bar{\nu} \pm \delta\bar{\nu}$ ($\delta\bar{\nu} \ll \bar{\nu}$), where the central component has twice the intensity of the other two components. Show that the intensity as a function of mirror separation becomes

$$I(x) = \frac{1}{2}I(0)\left[1 + \cos^2(2\pi\delta\bar{\nu}x)\cos(4\pi\bar{\nu}x)\right].$$

Sketch $I(x)$ and explain how you would determine $\delta\bar{\nu}/\bar{\nu}$ experimentally.
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PRELIMINARY EXAMINATION IN PHYSICS

MODERATIONS IN PHYSICS AND PHILOSOPHY

PRELIMINARY EXAMINATION IN PHYSICS AND PHILOSOPHY

SOME USEFUL DATA AND MATHEMATICAL FORMULAE

Abbreviations

A    ampere      Pa    pascal
C    coulomb     s    second  
eV   electron volt T    tesla
F    farad       V    volt
G    gramme      W    watt
H    henry       Wb   weber
Hz   hertz       mol  gramme mole
J    joule       K    kelvin
m    metre       Ω    ohm
N    newton      S    siemens (ohm⁻¹)

Prefixes

T    = tera = 10¹²   c    = centi = 10⁻²
G    = giga = 10⁹    m    = milli = 10⁻³
M    = mega = 10⁶    μ    = micro = 10⁻⁶
k    = kilo = 10³    n    = nano = 10⁻⁹
h    = hecto = 10²    p    = pico = 10⁻¹²
                     f    = femto = 10⁻¹⁵

Revised April 2011
Fundamental constants

Constants are given to 4 or 5 significant figures.

- **Electron rest mass**: \( m_e = 9.109 \times 10^{-31} \text{ kg} \)
- **Proton rest mass**: \( M_p = 1.6726 \times 10^{-27} \text{ kg} \)
- **Electronic charge**: \( e = 1.6022 \times 10^{-19} \text{ C} \)
- **Speed of light in free space**: \( c = 2.9979 \times 10^8 \text{ m s}^{-1} \)
- **Planck’s constant**: \( \hbar = 6.626 \times 10^{-34} \text{ J s} \)
  \( \hbar/2\pi = \hbar = 1.0546 \times 10^{-34} \text{ J s} \)
  \( \hbar c = 197.33 \text{ MeV fm} \)
- **Boltzmann’s constant**: \( k_B = 1.3807 \times 10^{-23} \text{ J K}^{-1} \)
- **Molar gas constant**: \( R = 8.315 \text{ J mol}^{-1} \text{ K}^{-1} \)
- **Avogadro’s number**: \( N = 6.022 \times 10^{23} \text{ mol}^{-1} \)
- **Standard molar volume**: \( V = 22.414 \times 10^{-3} \text{ m}^3 \text{ mol}^{-1} \)
- **Bohr magneton**: \( \mu_B = 9.274 \times 10^{-24} \text{ A m} \text{ or } \text{ J T}^{-1} \)
- **Nuclear magneton**: \( \mu_N = 5.051 \times 10^{-27} \text{ A m} \text{ or } \text{ J T}^{-1} \)
- **Bohr radius**: \( a_0 = 5.292 \times 10^{-11} \text{ m} \)
- **Fine structure constant**: \( e^2/(4\pi\epsilon_0\hbar c) = \alpha = (137.036)\text{ }^{-1} \)
- **Thomson cross section**: \( \sigma_T = 6.6524 \times 10^{-29} \text{ m}^2 \)
- **Compton wavelength of electron**: \( \hbar/(m_e c) = \lambda_C = 2.4263 \times 10^{-12} \text{ m} \)
- **Rydberg’s constant**: \( R_{\infty} = 1.0974 \times 10^7 \text{ m}^{-1} \)
  \( R_{\infty} \hbar c = 13.606 \text{ eV} \)
- **Stefan’s constant**: \( \sigma = 5.6704 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4} \)
- **Gravitational constant**: \( G = 6.6742 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2} \)
- **Proton magnetic moment**: \( \mu_p = 2.7928 \mu_N \)
- **Neutron magnetic moment**: \( \mu_n = -1.9130 \mu_N \)

\[ \text{Rest masses of some leptons and hadrons in MeV/e}^2 \text{ (the particle is denoted by its conventional symbol):} \]
\[ e^\pm 0.5110, \mu^\pm 105.66, \tau^\pm 1777, \pi^0 134.98, \pi^\pm 139.57, K^\pm 493.7, K^0 497.7, \eta 547, \]
\[ D^0 1865, \overline{D}^0 1870, \rho 938.5, \n 939.6, \Lambda^0 1116, \Sigma^+ 1189, \Sigma^0 1193, \Sigma^- 1197, \]
\[ \Xi^0 1315, \Xi^- 1321, \Omega^- 1672, Z^0 91.188 \times 10^3, W^\pm 80.40 \times 10^3, B^0 5280, B^\pm 5279. \]

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<th>Charge</th>
<th>( I_3 )</th>
<th>( S )</th>
<th>( C )</th>
<th>( B )</th>
<th>( T )</th>
<th>Mass (GeV/e^2)</th>
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<td>-( \frac{1}{2} )</td>
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<td>c</td>
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**Astrophysical data**

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<td>1 astronomical unit AU</td>
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<td>1 parsec pc</td>
<td>$3.086\times10^{16}$ m</td>
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<td>Radius of Sun $R_\odot$</td>
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**Geophysical data**

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<td>Molar heat capacity of dry air</td>
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<td>at constant pressure</td>
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<tr>
<td>at constant volume</td>
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</tr>
<tr>
<td>1 standard atmosphere</td>
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<tr>
<td>Acceleration due to gravity</td>
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<td>Solar constant</td>
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**Other data and conversion factors**

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<tr>
<td>1 fermi fm</td>
<td>$10^{-15}$ m</td>
</tr>
<tr>
<td>1 barn b</td>
<td>$10^{-28}$ m^2</td>
</tr>
<tr>
<td>1 cm^{-1}</td>
<td>$10^2$ m^{-1}</td>
</tr>
<tr>
<td>1 pascal Pa</td>
<td>$1\text{ N m}^{-2}$</td>
</tr>
<tr>
<td>Permeability of free space $\mu_0$</td>
<td>$4\pi \times 10^{-7}$ H m^{-1}</td>
</tr>
<tr>
<td>Permittivity of free space $\varepsilon_0$</td>
<td>$8.854 \times 10^{-12}$ F m^{-1}</td>
</tr>
<tr>
<td>1 electron volt</td>
<td>$1.6022 \times 10^{-19}$ J</td>
</tr>
<tr>
<td>$eV/\hbar c$</td>
<td>$8.065 \times 10^{5}$ m^{-1}</td>
</tr>
<tr>
<td>$eV/k_B$</td>
<td>$1.1604 \times 10^4$ K</td>
</tr>
<tr>
<td>1 unified atomic mass unit $^{12}$C</td>
<td>$931.5\text{ MeV/c}^2 = 1.6606 \times 10^{-27}$ kg</td>
</tr>
<tr>
<td>Wavelength of 1 eV photon</td>
<td>$1.23984 \times 10^{-6}$ m</td>
</tr>
</tbody>
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**Trigonometrical identities**

\[
\sin(\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi
\]
\[
\cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi
\]
\[
\sin \alpha + \sin \beta = 2 \sin \frac{1}{2} (\alpha + \beta) \cos \frac{1}{2} (\alpha - \beta)
\]
\[
\cos \alpha + \cos \beta = 2 \cos \frac{1}{2} (\alpha + \beta) \cos \frac{1}{2} (\alpha - \beta)
\]
\[
\cos \alpha - \cos \beta = 2 \sin \frac{1}{2} (\alpha + \beta) \sin \frac{1}{2} (\beta - \alpha)
\]

In a triangle ABC:

\[
\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}
\]
\[
a^2 = b^2 + c^2 - 2bc \cos A
\]
Vector identities

In a right handed system

\[ a \times b = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \]

Vector triple product

\[ a \times (b \times c) = (a.c)b - (a.b)c \]

If \( \varphi, U \) are scalar fields and \( \mathbf{F}, \mathbf{G} \) are vector fields:

- \( \text{curl}\, \text{curl}\, \mathbf{F} = \text{grad}\, \text{div}\, \mathbf{F} - \nabla^2 \mathbf{F} \)
- \( \text{div}(\varphi \mathbf{F}) = \varphi \text{grad}\, \varphi + \varphi \text{ div}\, \mathbf{F} \)
- \( \text{curl}(\varphi \mathbf{F}) = \varphi \text{ curl}\, \mathbf{F} - \mathbf{F} \times \text{grad} \varphi \)
- \( \text{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \text{curl}\, \mathbf{F} - \mathbf{F} \cdot \text{curl}\, \mathbf{G} \)
- \( \text{grad}(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times \text{curl}\, \mathbf{G} + \mathbf{G} \times \text{curl}\, \mathbf{F} \)
- \( \text{curl}(\mathbf{F} \times \mathbf{G}) = (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G} + \mathbf{F} \times \text{grad} \mathbf{G} - \mathbf{G} \times \text{grad} \mathbf{F} \)

\[
\iiint (U \nabla^2 \varphi - \varphi \nabla^2 U) dV = \iiint (U \text{grad} \varphi - \varphi \text{grad} U) dS
\]

Cylindrical coordinates:

\( \nabla^2 U = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2} \)

\( \text{grad} U = \left( \frac{\partial U}{\partial r}, \frac{1}{r} \frac{\partial U}{\partial \theta}, \frac{\partial U}{\partial z} \right) \)

Spherical polar coordinates:

\( \nabla^2 U = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial U}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial U}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 U}{\partial \phi^2} \)

\( \text{grad} U = \left( \frac{\partial U}{\partial r}, \frac{1}{r} \frac{\partial U}{\partial \theta}, \frac{1}{r \sin \theta} \frac{\partial U}{\partial \phi} \right) \)

Note that:

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial U}{\partial r} \right) = \frac{1}{r} \frac{\partial^2}{\partial r^2} (rU)
\]

Indefinite and definite integrals

Indefinite (with \( a > 0 \)):

\[
\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} \quad \int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1} \frac{x}{a}
\]

\[
\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right| \quad \int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a} \quad \text{if } x > a
\]

\[
\int \frac{dx}{\sqrt{a^2 - x^2}} = -\cosh^{-1} \frac{-x}{a} \quad \text{if } x < -a
\]

Definite:

\[
\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{m - 1}{m + n} \int_0^{\pi/2} \sin^{m-2} x \cos^n x \, dx = \frac{n - 1}{m + n} \int_0^{\pi/2} \sin^m x \cos^{n-2} x \, dx
\]

\[
I_n = \int_0^{\infty} x^n e^{-ax^2} \, dx \quad I_0 = \frac{\sqrt{\pi}}{2a}, \quad I_1 = \frac{1}{2a}, \quad I_{n+2} = \frac{(n+1)I_n}{2a} \quad \int_0^{\infty} x^{1/2} e^{-x} \, dx = 2.32
\]