A7 Numerical Analysis

TRINITY TERM: 1.5 hours

- You may hand in attempts to any number of questions.
- Answers to the two best questions will count towards the total mark for the paper.
- All questions are worth 25 marks.

Do not turn this page until you are told that you may do so
1. Suppose that \( n \) is a positive integer, and let \( \Pi_n \) denote the set of all polynomials of degree less than or equal to \( n \) with real coefficients. Suppose further that \( f \) is a real-valued function, defined and continuous on a nonempty closed interval \([a, b]\) of \( \mathbb{R} \), and consider \( n + 1 \) real numbers \( x_i \in [a, b] \), \( i = 0, 1, \ldots, n \), such that \( x_i \neq x_j \) when \( i \neq j \).

(a) [8 marks] What does it mean to say that \( p_n \in \Pi_n \) interpolates \( f \) at \( x_i \), \( i = 0, 1, \ldots, n \)?

Show that there exists a polynomial \( p_n \in \Pi_n \), called the Lagrange interpolation polynomial of degree \( n \) for \( f \), that interpolates \( f \) at \( x_i \), \( i = 0, 1, \ldots, n \).

Show further that the Lagrange interpolation polynomial of degree \( n \) for \( f \) with interpolation points \( x_i \), \( i = 0, 1, \ldots, n \), is the unique polynomial in \( \Pi_n \) that interpolates \( f \) at the points \( x_i \), \( i = 0, 1, \ldots, n \).

(b) [5 marks] Deduce from part (a) that the matrix

\[
A_n = \begin{pmatrix}
1 & x_0 & x_0^2 & \ldots & x_0^n \\
1 & x_1 & x_1^2 & \ldots & x_1^n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x_n & x_n^2 & \ldots & x_n^n
\end{pmatrix}
\]

is invertible.

(c) [5 marks] Now suppose that \( f^{(n+1)} \), the \((n+1)\)-th derivative of \( f \), is defined and continuous on the interval \([a, b]\). Show that for each \( x \in [a, b] \) there exists a real number \( \xi = \xi(x) \in (a, b) \) such that

\[
f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i),
\]

where \( p_n \) is the Lagrange interpolation polynomial of degree \( n \) for \( f \) with interpolation points \( x_i \), \( i = 0, 1, \ldots, n \).

(d) [7 marks] Suppose that \( f(x) = \sin(2x), x \in [0, 1] \), and let \( p_n \) denote the Lagrange interpolation polynomial of degree \( n \) for \( f \) with interpolation points \( x_i = i/n, i = 0, 1, \ldots, n \).

Show that

\[
\max_{x \in [0, 1]} |f(x) - p_n(x)| \leq \frac{2^{[n+1]}}{(n+1)!},
\]

where, for a positive real number \( s \), \([s]\) denotes the integer part of \( s \). Hence deduce that the sequence of Lagrange interpolation polynomials \( \{p_n\}_{n=1}^{\infty} \) converges to \( f \) uniformly on the interval \([0, 1] \).
2. Suppose that $n \geq 2$, $v \in \mathbb{R}^n$, $v \neq 0$, and $\alpha \in \mathbb{R}$. Consider the $n \times n$ matrix $H_{v,\alpha}$ defined by
\[ H_{v,\alpha} := I_n - \frac{\alpha}{v^TV}vv^T, \]
where $I_n$ is the $n \times n$ identity matrix.

(a) [4 marks] Show that $H_{v,\alpha}$ is a symmetric matrix. Show further that $H_{v,\alpha}$ is an orthogonal matrix if, and only if, $\alpha \in \{0, 2\}$.

(b) [4 marks] When $\alpha = 2$ the matrix $H_v := H_{v,2}$ is referred to as a Householder matrix. Define $S_v := \{x \in \mathbb{R}^n : v^Tx = 0\}$. Show that $H_vx = x$ for all $x \in S_v$. Show further that $v^TH_vx = -v^Tx$ for all $x \in \mathbb{R}^n$.

(c) [8 marks] Show that for any $x \in \mathbb{R}^n$, $x \neq 0$, there exists $v \in \mathbb{R}^n$, $v \neq 0$ and $c \in \mathbb{R}$, $c \neq 0$, such that $H_vx = ce_1$, where $e_1 = (1, 0, \ldots, 0)^T \in \mathbb{R}^n$.

(d) [9 marks] Suppose that $n \geq 3$ and $A$ is a symmetric $n \times n$ matrix with real entries. Show that there exists an $n \times n$ orthogonal matrix $Q$ such that $Q^TAQ$ is a tridiagonal matrix.

3. (a) [16 marks] What is the fundamental row operation used repeatedly in Gaussian Elimination (GE)? How can this operation on a matrix $A = \{a_{i,j} : i, j = 1, \ldots, n\}$ be expressed as premultiplication of $A$ by another matrix?

Hence express GE applied to a matrix $A \in \mathbb{R}^{n \times n}$ with $n = 4$ as a sequence of such premultiplications.

Deduce that for general $n$ the number of such premultiplications is $n(n - 1)/2$.

What matrix factorisation does this lead to? [You should explain why, and describe the form of the matrix factors.]

(b) [9 marks] Suppose now that the matrix $A = \{a_{i,j} : i, j = 1, \ldots, n\} \in \mathbb{R}^{n \times n}$ is known to be such that $a_{i,j} = 0$ whenever $j < i - 1$; how might the above factorisation be shortened and what will be the forms of the matrix factors in this case?

If in fact $A$ is also symmetric, deduce that both matrix factors have at most 2 non-zero entries in each row.

[Throughout this question you should assume that no row interchanges are required.]