SECTION A

Attempt ALL questions.

1. If \( z = 1 + i \), determine the real and imaginary parts of \( iz \), \( z^{100} \), and \( e^z \). [5]

2. Expand \( f(x) = \cos x \) as a power series about \( x = \pi/4 \), up to and including the first three non-zero terms. [5]

3. Find the particular solution of the initial value problem
   
   \[
   \frac{dy}{dx} = \frac{2(x + 2y)}{x - y} \quad \text{where} \quad y(1) = 0 .
   \]

   [5]

4. Calculate the directional derivative of \( f(x, y, z) = x^2 + y^2 + z^2 \) in the direction of \( i - j \), at the point \( (x, y, z) = (0, 2, 0) \). [5]

5. If
   
   \[
   \sin(x^2 + y^2) = y^2 + z^2 ,
   \]

   determine
   
   \[
   \left( \frac{\partial x}{\partial y} \right)_z \quad \text{and} \quad \left( \frac{\partial y}{\partial x} \right)_z .
   \]

   [5]

6. Calculate

   \[
   \iint_{R} x^2 \, dx \, dy
   \]

   where \( R \) is the unit disk \( x^2 + y^2 \leq 1 \). [5]

7. The Bessel functions \( J_0 \) and \( J_1 \) satisfy the following relations

   \[
   \frac{d}{ds} \left( sJ_1(s) \right) = sJ_0(s) \quad \text{and} \quad \frac{d}{ds} J_0(s) = -J_1(s) .
   \]

   For \( n = 2, 3, 4, \ldots \), derive the reduction formula

   \[
   \int_{0}^{x} s^n J_0(s) \, ds = x^n J_1(x) + (n - 1)x^{n-1}J_0(x) - (n - 1)^2 \int_{0}^{x} s^{n-2} J_0(s) \, ds .
   \]  [5]

8. If the vectors \( a, b \) and \( c \) satisfy

   \[
   a + b + c = 0 ,
   \]

   show that

   \[
   a \times b = b \times c = c \times a .
   \]  [5]
SECTION B

Attempt THREE questions.

9. (a) Define the terms even and odd functions. Explain how symmetry can simplify the process of integration. [5]

(b) The density of a sheet of material at coordinates \((x, y)\) is

\[ \rho(x, y) = x^2. \]

(i) From the sheet of material, a rectangular lamina \(R_1\) with diagonal corners at \((x, y) = (-1, 0)\) and \((1, 1)\) is cut out, as shown below. Find the mass and the centre of mass of this lamina. [4]

(ii) A second lamina, \(R_2\), in the shape of half a disk of radius 2, is cut out from the same sheet, as shown below. Find the centre of mass of this lamina. [4]

(c) A rectangular lamina, of the same shape as \(R_1\) above, is cut from a different piece of material with density

\[ \rho(x, y) = y \exp(-x^4). \]

Find its centre of mass. [4]

(d) From a third piece of material, two laminas are cut in the shapes of \(R_1\) and \(R_2\) as above. The masses of the laminas are \(M_1\) and \(M_2\), respectively, and the \(y\)-coordinates of their centres of mass are \(y_1\) and \(y_2\).

Show that the combined lamina, \(R_1 + R_2\), has a centre of mass with \(y\)-coordinate

\[ \frac{M_1 y_1 + M_2 y_2}{M_1 + M_2}. \] [3]

Turn over.
10. (a) What is meant by a linear differential equation?

(b) The functions $T(\theta)$ and $R(r)$ are solutions of the differential equations

$$\frac{d^2T}{d\theta^2} + m^2 T = 0 \quad (1)$$

$$r^2 \frac{d^2R}{dr^2} + r \frac{dR}{dr} - m^2 R = 0 \quad (2)$$

where $m$ is a constant.

Confirm that $f(r, \theta) = R(r)T(\theta)$ satisfies Laplace’s equation in polar coordinates,

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \left( \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} = 0,$$

for any value of $m$, assuming $r > 0$.

(c) Find the general solution of equation (1) for $T(\theta)$. What does imposing the boundary condition $T(\theta + 2\pi) = T(\theta)$ require?

(d) By assuming a trial solution of the form $r^n$, determine the general solution of equation (2) when $m \neq 0$, explaining your reasoning.

(e) Explain why the method of part (d) fails when $m = 0$.

Solve equation (2) in the case of $m = 0$, by transforming it into a first-order differential equation, or otherwise.
11. A solid material in a magnetic field $B$ at temperature $T$ has an internal energy $U(B, T)$, and an entropy $S(B, T)$.

(a) Explain why $U$ can alternatively be regarded as a function of $B$ and $S$. \[1\]

(b) In the light of part (a), and by considering the total differentials $dU$ and $dS$, show that
\[
\left( \frac{\partial U}{\partial B} \right)_T = \left( \frac{\partial U}{\partial B} \right)_S + \left( \frac{\partial U}{\partial S} \right)_B \left( \frac{\partial S}{\partial B} \right)_T. \tag{6}
\]

(c) Thermodynamic arguments imply that
\[
dU = TdS - MdB,
\]
where $M$ is the magnetization of the material. By considering the Helmholtz free energy, defined by $A = U - TS$, obtain the Maxwell relation
\[
\left( \frac{\partial M}{\partial T} \right)_B = \left( \frac{\partial S}{\partial B} \right)_T.
\]
Hence show that
\[
\left( \frac{\partial U}{\partial B} \right)_T = T \left( \frac{\partial M}{\partial T} \right)_B - M,
\]
and determine $\left( \frac{\partial U}{\partial B} \right)_T$ for a material that obeys Curie’s Law,
\[
M = \frac{CB}{T}, \tag{8}
\]
where $C$ is a constant.

(d) An ideal paramagnet has a magnetization of the form
\[
M = N\mu \tanh \left( \frac{\mu B}{kT} \right),
\]
where $N$, $\mu$ and $k$ are constants.
Using a Taylor expansion, show that the paramagnet obeys Curie’s Law when $\mu B \ll kT$. Determine $M$ as $\mu B/(kT) \to \infty$. \[5\]
12. (a) The probability density function for the exponential distribution takes the form

\[ f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases} \]

where \( \lambda \) and \( N \) are positive constants.

(i) Determine \( N \) such that \( f(x) \) is normalized. \[ 2 \]

(ii) Find the mean and standard deviation of the distribution. \[ 6 \]

(b) Consider a function of \( x \) and \( y \) defined by

\[ g(x, y) = ax^2 + 2bxy + cy^2. \]

The quantities \( a, b \) and \( c \) are constants that can be assumed to be real and non-zero.

(i) What is meant by a stationary point of the function \( g(x, y) \)? Obtain a pair of simultaneous equations to solve for the positions of these stationary points. \[ 2 \]

(ii) Taking your equations from part (i), find condition(s) on \( a, b \) and \( c \) such that the equations have non-trivial solutions for \( x \) and \( y \). \[ 2 \]

(iii) Assume that \( a, b \) and \( c \) do not satisfy the condition(s) identified in part (ii). Locate and classify any stationary points of \( g(x, y) \), expressing your answer in terms of \( a, b \) and \( c \). \[ 4 \]

(iv) Assume now that \( a, b \) and \( c \) do satisfy the condition(s) identified in part (ii). Locate the stationary points of \( g \), and classify them by ‘completing the square’ in \( g \), or otherwise. \[ 4 \]
13. Let $a > 0$ be a real number. The nonzero complex number $u$ satisfies $|u - a| = a$. All the possible values of $u$ are indicated by the circle of radius $a$ shown on the accompanying Argand diagram.

(a) The nonzero complex number $v$ satisfies

$$|v + a| = a.$$  \hspace{1cm} (1)

Sketch a copy of the Argand diagram above and include the curve that illustrates all the possible values of $v$ in the complex plane. \hspace{1cm} [3]

(b) Suppose that $v = \lambda u$ where $\lambda \neq 0$ is a real number. By eliminating $v$ from equation (1) in part (a), show that

$$\lambda = \frac{2a \text{Im}(u)}{|u|^2}.$$  \hspace{1cm} [5]

(c) On your diagram of part (a), indicate a typical value of $u$ and the corresponding point $v$. \hspace{1cm} [3]

(d) In terms of $a$, calculate

$$|u|^2 + |v|^2 \text{ and } |u - v|^2.$$  \hspace{1cm} [6]

(e) By referring to your diagram of part (a), explain the geometric significance of the result in part (d). \hspace{1cm} [3]

----------- End of Examination Paper -----------
Table of derivatives

<table>
<thead>
<tr>
<th>Function</th>
<th>Derivative</th>
<th>Function</th>
<th>Derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>( \frac{dy}{dx} )</td>
<td>( y )</td>
<td>( \frac{dy}{dx} )</td>
</tr>
<tr>
<td>( ax^n )</td>
<td>( anx^{n-1} )</td>
<td>( \exp x )</td>
<td>( \exp x )</td>
</tr>
<tr>
<td>( \sin x )</td>
<td>( \cos x )</td>
<td>( \sin x )</td>
<td>( \cosh x )</td>
</tr>
<tr>
<td>( \cos x )</td>
<td>( -\sin x )</td>
<td>( \cosh x )</td>
<td>( \sinh x )</td>
</tr>
<tr>
<td>( \tan x )</td>
<td>( \sec^2 x )</td>
<td>( \tanh x )</td>
<td>( \sech^2 x )</td>
</tr>
<tr>
<td>( \sec x )</td>
<td>( \sec x \tan x )</td>
<td>( \sech x )</td>
<td>( -\sech x \tanh x )</td>
</tr>
<tr>
<td>( \cosec x )</td>
<td>( -\cosec x \cot x )</td>
<td>( \coth x )</td>
<td>( -\coth x \coth x )</td>
</tr>
<tr>
<td>( \cot x )</td>
<td>( -\cosec^2 x )</td>
<td>( \coth x )</td>
<td>( -\coth x \coth x )</td>
</tr>
<tr>
<td>( \sin^{-1} \left( \frac{x}{a} \right) )</td>
<td>( \pm \frac{1}{\sqrt{a^2 - x^2}} )</td>
<td>( \sinh^{-1} \left( \frac{x}{a} \right) )</td>
<td>( \frac{1}{\sqrt{x^2 + a^2}} )</td>
</tr>
<tr>
<td>( \cos^{-1} \left( \frac{x}{a} \right) )</td>
<td>( \frac{1}{\sqrt{a^2 - x^2}} )</td>
<td>( \cosh^{-1} \left( \frac{x}{a} \right) )</td>
<td>( \frac{1}{\sqrt{x^2 + a^2}} )</td>
</tr>
<tr>
<td>( \tan^{-1} \left( \frac{x}{a} \right) )</td>
<td>( \frac{a}{x^2 + a^2} )</td>
<td>( \tanh^{-1} \left( \frac{x}{a} \right) )</td>
<td>( \frac{a}{(a^2 - x^2)} )</td>
</tr>
<tr>
<td>( \log x )</td>
<td>( \frac{1}{x} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: \( a \) is a constant

Table of standard integrals

<table>
<thead>
<tr>
<th>Function</th>
<th>Integral</th>
<th>Function</th>
<th>Integral</th>
</tr>
</thead>
<tbody>
<tr>
<td>( ax^n )</td>
<td>( \frac{ax^{n+1}}{n+1}; n \neq -1 )</td>
<td>( \exp (ax) )</td>
<td>( \frac{1}{a} \exp (ax) )</td>
</tr>
<tr>
<td>( \frac{1}{x} )</td>
<td>( \log</td>
<td>x</td>
<td>)</td>
</tr>
<tr>
<td>( \sin x )</td>
<td>( -\cos x )</td>
<td>( \cosh x )</td>
<td>( \sinh x )</td>
</tr>
<tr>
<td>( \cos x )</td>
<td>( \sin x )</td>
<td>( \tanh x )</td>
<td>( \log</td>
</tr>
<tr>
<td>( \sec x )</td>
<td>( \log</td>
<td>\sec x + \tan x</td>
<td>)</td>
</tr>
<tr>
<td>( \cosec x )</td>
<td>( -\log</td>
<td>\cosec x + \cot x</td>
<td>)</td>
</tr>
<tr>
<td>( \cot x )</td>
<td>( \log</td>
<td>\sin x</td>
<td>)</td>
</tr>
<tr>
<td>( \sec^2 x )</td>
<td>( \tan x )</td>
<td>( \cosech^2 x )</td>
<td>( -\coth x )</td>
</tr>
<tr>
<td>( \cosec^2 x )</td>
<td>( -\cot x )</td>
<td>( \sech x \tanh x )</td>
<td>( -\sech x )</td>
</tr>
<tr>
<td>( \sec x \tan x )</td>
<td>( \sec x )</td>
<td>( \cosech x \coth x )</td>
<td>( -\cosech x )</td>
</tr>
<tr>
<td>( \cosec x \cot x )</td>
<td>( -\cosec x )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \frac{1}{\sqrt{a^2 - x^2}} )</td>
<td>( \sin^{-1} \left( \frac{x}{a} \right) )</td>
<td>( \frac{1}{\sqrt{x^2 + a^2}} )</td>
<td>( -\tan^{-1} \left( \frac{x}{a} \right) )</td>
</tr>
<tr>
<td>( \frac{1}{x^2 + a^2} )</td>
<td>( \frac{1}{a} )</td>
<td>( \frac{1}{\sqrt{x^2 + a^2}} )</td>
<td>( \sinh^{-1} \left( \frac{x}{a} \right) )</td>
</tr>
<tr>
<td>( \frac{1}{\sqrt{x^2 - a^2}} )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\begin{cases}
    \cosh^{-1} \left( \frac{x}{a} \right) & x > |a| \\
    -\cosh^{-1} \left( \frac{-x}{a} \right) & x < -|a| 
\end{cases}
\]

Note: \( a \) is a constant and the constant of integration has been omitted.
Trigonometry

Compound angles

\[
\begin{align*}
sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\
tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}
\end{align*}
\]

Half Angle

\[
\begin{align*}
\sin A &= \frac{2t}{1 + t^2} \\
\cos A &= \frac{1 - t^2}{1 + t^2} \\
\tan A &= \frac{2t}{1 - t^2}
\end{align*}
\]

In which \( t = \tan \frac{A}{2} \)

Sum and difference formulae

\[
\begin{align*}
\sin A + \sin B &= 2 \sin \left(\frac{A + B}{2}\right) \cos \left(\frac{A - B}{2}\right) \\
\sin A - \sin B &= 2 \cos \left(\frac{A + B}{2}\right) \sin \left(\frac{A - B}{2}\right) \\
\cos A + \cos B &= 2 \cos \left(\frac{A + B}{2}\right) \cos \left(\frac{A - B}{2}\right) \\
\cos A - \cos B &= -2 \sin \left(\frac{A + B}{2}\right) \sin \left(\frac{A - B}{2}\right)
\end{align*}
\]