1. Explain what is meant by a function of state. Give an expression for the change in entropy when a body is heated between temperatures $T_1$ and $T_2$. [5]

2. Consider the Carnot cycle for an ideal gas operating between temperatures $T_1$ and $T_2$.

The figure shows the cycle’s entropy-temperature diagram. Show that the efficiency of the cycle is

$$\eta = 1 - \frac{T_1}{T_2}.$$ 

Hence estimate the maximum efficiency possible for a practical steam engine when the steam is heated to 800 K. [7]

3. Gas with thermal conductivity $\kappa$ fills the space between two coaxial cylinders (inner cylinder radius $a$, outer cylinder inner radius $b$). A current $I$ is passed through the inner cylinder, which has resistivity $\rho$. Derive an expression for the equilibrium temperature of the inner cylinder $T_a$ when the outer cylinder is held at a constant temperature $T_b$. [8]

4. A vessel holding some liquid is placed in a vacuum chamber that is constantly pumped. Gas leaks from the vessel into the vacuum through a small hole, radius 4.15 $\mu$m, but the pressure inside the vessel remains 38.6 kPa. After a day the mass of the vessel has dropped from 100 g to 99.5 g. Assuming the container has at all times been at 70 K, suggest (with reasoning) the identity of the liquid. [6]
5. Show that the mean-scattering time of a molecule travelling at speed \( v \) is

\[
\tau = \frac{1}{n \sigma v},
\]

where \( n \) is the molecular density and \( \sigma \) is the collision cross section. Estimate the mean time between collisions of nitrogen molecules at sea level in the Earth’s atmosphere assuming they are travelling at their mean speed. [8]

[You may assume nitrogen molecules are spheres of diameter 0.37 nm and the atmosphere is pure nitrogen.]

6. The isothermal compressibility \( \kappa_T \) and the adiabatic compressibility \( \kappa_S \) are defined by

\[
\kappa_T = -\frac{1}{V} \left( \frac{\partial V}{\partial p} \right)_T \quad \text{and} \quad \kappa_S = -\frac{1}{V} \left( \frac{\partial V}{\partial p} \right)_S,
\]

where the symbols have their usual meanings. Show that \( \kappa_S/\kappa_T = 1/\gamma \) where \( \gamma \) is the ratio of principal specific heats. [6]

Section B

7. The coefficient of viscosity \( \eta \) is defined by

\[
\tau_{xz} = \eta \frac{\partial u_x}{\partial z}.
\]

Explain the terms in this equation. [4]

Using simple kinetic theory, show that the viscosity of a gas is

\[
\eta = \frac{1}{3} n m \lambda \langle v \rangle,
\]

where \( n \) is the molecular number density, \( m \) the molecular mass, \( \lambda \) the mean free path and \( \langle v \rangle \) is the mean molecular velocity. [6]

A square read-head of area 1 m\(^2\) is suspended above a disk rotating at 6000 rpm. The height of the read-head above the disk is 25 \( \mu \)m. Assuming the velocity gradient of the air between the head and the disk is linear, what viscous force does the head experience when it is positioned 3 cm from the axis of rotation? [10]

[Assume the viscosity of air \( \eta_{\text{air}} = 1.73 \times 10^{-5} \text{ N s m}^{-2} \).]
8. Define the partition function $Z$ for a system in terms of the energies $E_j$ of its quantum states $j$ and the inverse temperature $\beta = (k_B T)^{-1}$. Write down the probability $p_j$ that the system is in a given state $j$.

Show that the system’s internal energy $U$, entropy $S = -k_B \sum_j p_j \log_e p_j$ and Helmholtz free energy $F$ are given by

$$U = -\frac{\partial \log_e Z}{\partial \beta}, \quad S = \frac{U}{T} + k_B \log_e Z \quad \text{and} \quad F = -k_B T \log_e Z.$$  

In a simplified model of a crystal, each molecule is a point mass that is attached to its site by a force, so that at each site there is a three-dimensional quantised harmonic oscillator with natural angular frequency $\omega$. Show that in this approximation the Helmholtz free energy of a crystal of $N$ sites is

$$F = \frac{3}{2} N \hbar \omega + 3N k_B T \log_e \left(1 - e^{-\hbar \omega / k_B T}\right).$$

Show that the crystal’s heat capacity (a) tends to $3N k_B$ in the limit of high temperature, $T \to \infty$, and (b) vanishes in the limit of low temperature, $T \to 0$.

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}, \quad \text{for} \quad |r| < 1.$$  

9. Show that the relation between pressure $p$ and temperature $T$ when two phases of a single substance are in equilibrium is given by the Clausius-Clapeyron equation,

$$\frac{dp}{dT} = \frac{\Delta S}{\Delta V} = \frac{L}{T\Delta V},$$

where $\Delta S$ is the change in specific entropy for a given change in specific volume $\Delta V$ when the substance changes phase, and where $L$ is the specific latent heat.

A pressurised vessel containing liquid water and water vapour only is heated until the pressure reaches twice the value of atmospheric pressure. Obtain an expression for the specific volume of the vapour by assuming that it behaves as a perfect gas. Hence calculate the temperature at which this pressure will be reached. You may assume the latent heat to be independent of temperature, and that the specific volume of the water is negligible compared with that of the vapour.

Explain briefly why $dp/dT$ is negative when liquid water and water ice are in equilibrium.

[The specific latent heat of vaporisation of water is $2.272 \times 10^6 \text{ J kg}^{-1}$, the mean molecular mass of water is 18.0 a.m.u., and atmospheric pressure may be taken to be $10^5 \text{ Pa.}$.]
10. Use the method of separation of variables to show that the wave equation

\[ \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0 \]

has solutions of the form

\[ y = \begin{cases} 
\sin kx \cos kct, \\
\sin kx \sin kct, \\
\cos kx \cos kct, \\
\cos kx \sin kct.
\end{cases} \]

What is a *normal mode* of a string? State two properties of normal modes that make them useful in the analysis of oscillating systems. [3]

A string in which transverse displacements propagate at speed \( c \) has its ends fixed at \( x = 0 \) and \( x = \pi \). The midpoint of the string is drawn aside a small distance \( h \) and gently released. Show that at any subsequent time \( t \) the displacement is

\[ y = \frac{8h}{\pi^2} \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)^2} \sin(2r+1)x \cos(2r+1)ct. \] [10]

What is the ratio of the energies contained in the fundamental and third normal modes? [4]
**Fundamental constants**

Constants are given to 4 or 5 significant figures (data from NIST CODATA 2014).

<table>
<thead>
<tr>
<th>Physical Constant</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electron rest mass</td>
<td>$m_e = 9.109 \times 10^{-31}$ kg</td>
</tr>
<tr>
<td>Proton rest mass</td>
<td>$M_p = 1.6726 \times 10^{-27}$ kg</td>
</tr>
<tr>
<td>Electronic charge</td>
<td>$e = 1.6022 \times 10^{-19}$ C</td>
</tr>
<tr>
<td>Speed of light in free space</td>
<td>$c = 2.9979 \times 10^8$ m s$^{-1}$</td>
</tr>
<tr>
<td>Planck’s constant</td>
<td>$h = 6.626 \times 10^{-34}$ J s</td>
</tr>
<tr>
<td>$h/2\pi = \hbar$</td>
<td>$1.0546 \times 10^{-34}$ J s</td>
</tr>
<tr>
<td>$\hbar c$</td>
<td>$197.33$ MeV fm</td>
</tr>
<tr>
<td>Boltzmann’s constant</td>
<td>$k_B = 1.3806 \times 10^{-23}$ J K$^{-1}$</td>
</tr>
<tr>
<td>Molar gas constant</td>
<td>$R = 8.314$ J mol$^{-1}$ K$^{-1}$</td>
</tr>
<tr>
<td>Avogadro’s number</td>
<td>$N_A = 6.022 \times 10^{23}$ mol$^{-1}$</td>
</tr>
<tr>
<td>Standard molar volume</td>
<td>$22.414 \times 10^{-3}$ m$^3$ mol$^{-1}$</td>
</tr>
<tr>
<td>Bohr magneton</td>
<td>$\mu_B = 9.274 \times 10^{-24}$ A m$^2$ or J T$^{-1}$</td>
</tr>
<tr>
<td>Nuclear magneton</td>
<td>$\mu_N = 5.051 \times 10^{-27}$ A m$^2$ or J T$^{-1}$</td>
</tr>
<tr>
<td>Bohr radius</td>
<td>$a_0 = 5.292 \times 10^{-11}$ m</td>
</tr>
<tr>
<td>Fine structure constant</td>
<td>$e^2/(4\pi\varepsilon_0\hbar c) = \alpha$ $(137.04)^{-1}$</td>
</tr>
<tr>
<td>Thomson cross section</td>
<td>$\sigma_T = 6.6525 \times 10^{-29}$ m$^2$</td>
</tr>
<tr>
<td>Compton wavelength of electron</td>
<td>$h/(m_ec) = \lambda_C = 2.4263 \times 10^{-12}$ m</td>
</tr>
<tr>
<td>Rydberg’s constant</td>
<td>$R_\infty = 1.0974 \times 10^7$ m$^{-1}$</td>
</tr>
<tr>
<td>$R_\infty h c$</td>
<td>$13.606$ eV</td>
</tr>
<tr>
<td>Stefan’s constant</td>
<td>$\sigma = 5.6704 \times 10^{-8}$ W m$^{-2}$ K$^{-4}$</td>
</tr>
<tr>
<td>Gravitational constant</td>
<td>$G = 6.6741 \times 10^{-11}$ N m$^2$ kg$^{-2}$</td>
</tr>
<tr>
<td>Proton magnetic moment</td>
<td>$\mu_p = 2.7928 \mu_N$</td>
</tr>
<tr>
<td>Neutron magnetic moment</td>
<td>$\mu_n = -1.9130 \mu_N$</td>
</tr>
</tbody>
</table>

Revised November 2018
Other data and conversion factors

1 angstrom Å $10^{-10}$ m
1 fermi fm $10^{-15}$ m
1 barn b $10^{-28}$ m²
1 cm$^{-1}$ $10^2$ m$^{-1}$
1 pascal Pa $1$ N m$^{-2}$

Permeability of free space

$$\mu_0 = 4\pi \times 10^{-7} \text{ H m}^{-1}$$

Permittivity of free space

$$\varepsilon_0 = 8.854 \times 10^{-12} \text{ F m}^{-1}$$

1 electron volt

$$\text{eV} = 1.6022 \times 10^{-19} \text{ J}$$
$$\text{eV}/hc = 8.065 \times 10^5 \text{ m}^{-1}$$
$$\text{eV}/k_B = 1.604 \times 10^4 \text{ K}$$

1 unified atomic mass unit (12C scale)

$$\text{u} = 931.5 \text{ MeV/c}^2 = 1.6605 \times 10^{-27} \text{ kg}$$

Wavelength of 1 eV photon

$$\lambda = 1.23984 \times 10^{-6} \text{ m}$$

Particle physics data

Masses (the symbols have their conventional meanings) in MeV/c$^2$ (PDG 2018):

\begin{align*}
ed^\pm, \mu^\pm, \tau^\pm, \pi^0, \pi^\pm, K^\pm, K^0, \eta, D^0, D^\pm, p, n, \Lambda^0, \Sigma^+, \Sigma^0, \Sigma^-, \Xi^0, \Xi^-, \Omega^-, B^0, B^\pm, Z^0, W^\pm, H^0\end{align*}

<table>
<thead>
<tr>
<th>Quark</th>
<th>Charge</th>
<th>$I_3$</th>
<th>$S$</th>
<th>$C$</th>
<th>$B$</th>
<th>$T$</th>
<th>Mass (GeV/c$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>u</td>
<td>$+\frac{2}{3}$</td>
<td>$+\frac{1}{2}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>&lt; 1</td>
</tr>
<tr>
<td>d</td>
<td>$-\frac{1}{3}$</td>
<td>$-\frac{1}{2}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>&lt; 1</td>
</tr>
<tr>
<td>c</td>
<td>$+\frac{2}{3}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>+1</td>
<td>0</td>
<td>1.27</td>
</tr>
<tr>
<td>s</td>
<td>$-\frac{1}{3}$</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>&lt; 1</td>
</tr>
<tr>
<td>t</td>
<td>$+\frac{2}{3}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>+1</td>
<td>173.0</td>
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<tr>
<td>b</td>
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<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>4.18</td>
</tr>
</tbody>
</table>

Astrophysical data

| 1 astronomical unit AU | $1.496 \times 10^{11}$ m | Mean radius of Earth | 6371 km |
| 1 parsec pc | $3.086 \times 10^{16}$ m | Molar mass of dry air | $28.96 \times 10^{-3}$ kg |
| Luminosity of Sun $L_\odot$ | $3.828 \times 10^{26}$ W | Molar heat capacity of dry air | at constant pressure | $29.13 \text{ J K}^{-1}$ |
| Mass of Sun $M_\odot$ | $1.988 \times 10^{30}$ kg | at constant volume | $20.83 \text{ J K}^{-1}$ |
| Radius of Sun $R_\odot$ | $6.957 \times 10^8$ m | 1 standard atmosphere | $1.013 \times 10^5$ Pa |
| Solar constant | | Acceleration due to gravity | $9.807 \text{ m s}^{-2}$ |
| | | | $1361 \text{ W m}^{-2}$ |
Trigonometrical identities

\[
\sin(\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi \\
\cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi \\
\sin \alpha + \sin \beta = 2 \sin \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta) \\
\cos \alpha + \cos \beta = 2 \cos \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta) \\
\cos \alpha - \cos \beta = 2 \sin \frac{1}{2}(\alpha + \beta) \sin \frac{1}{2}(\beta - \alpha)
\]

In a triangle ABC:

\[
\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \\
a^2 = b^2 + c^2 - 2bc \cos A
\]

Indefinite and definite integrals

Indefinite (with \(a > 0\)):

\[
\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} \\
\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| \\
\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1} \frac{x}{a} \\
\int \frac{dx}{\sqrt{x^2 - a^2}} = \left\{ \begin{array}{ll} \cosh^{-1} \frac{x}{a} & \text{if } x > a \\ -\cosh^{-1} \frac{-x}{a} & \text{if } x < -a \end{array} \right. \\
\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}
\]

Definite:

\[
\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{m-1}{m+n} \int_0^{\pi/2} \sin^{m-2} x \cos^n x \, dx \\
\int_0^{\pi/2} \sin^m x \cos^{n-2} x \, dx
\]

\[
I_n = \int_0^\infty x^n e^{-ax^2} \, dx : \ I_0 = \frac{1}{2} \sqrt{\frac{\pi}{a}} , \ I_1 = \frac{1}{2a} , \ I_{n+2} = \frac{(n+1)I_n}{2a} ; \ \int_0^\infty \frac{x^{1/2}}{\sqrt{e^x - 1}} \, dx = 2.32
\]

Vector identities

In a right handed system \( \mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \)

Vector triple product \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \)
Vector calculus

If $U$ is a scalar field and $F$ is a vector field:

Cartesian coordinates:
\[ \nabla U = \text{grad} U = \left( \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} \right) \]
\[ \nabla \cdot F = \text{div} F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \]
\[ \nabla \times F = \text{curl} F = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \]
\[ \nabla^2 U = \nabla \cdot (\nabla U) = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \]
\[ \nabla^2 F = (\nabla^2 F_x, \nabla^2 F_y, \nabla^2 F_z) \]

Cylindrical coordinates:
\[ \nabla U = \left( \frac{\partial U}{\partial r}, \frac{1}{r} \frac{\partial U}{\partial \phi}, \frac{\partial U}{\partial z} \right) \]
\[ \nabla \cdot F = \frac{1}{r} \frac{\partial}{\partial r} \left( r F_r \right) + \frac{1}{r} \frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z} \]
\[ \nabla \times F = \left( \frac{1}{r} \frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \]
\[ \nabla^2 U = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2} \]

Spherical polar coordinates:
\[ \nabla U = \left( \frac{\partial U}{\partial r}, \frac{1}{r} \frac{\partial U}{\partial \theta}, \frac{1}{r \sin \theta} \frac{\partial U}{\partial \phi} \right) \]
\[ \nabla \cdot F = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 F_r \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta F_\theta \right) + \frac{1}{r \sin \theta} \frac{\partial F_\phi}{\partial \phi} \]
\[ \nabla \times F = \left( \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} \sin \theta F_\phi - \frac{\partial F_\theta}{\partial \phi} \right], \frac{1}{r} \frac{\partial (r F_\theta)}{\partial \phi} - \frac{\partial F_r}{\partial \theta}, \frac{1}{r} \frac{\partial (r F_\phi)}{\partial \theta} - \frac{\partial F_r}{\partial \phi} \right) \]
\[ \nabla^2 U = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial U}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial U}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 U}{\partial \phi^2} \]

Note that:
\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial U}{\partial r} \right) \equiv \frac{1}{r} \frac{\partial^2}{\partial r^2} (rU) \]

Useful identities:
\[ \nabla \times (\nabla \times F) = \nabla (\nabla \cdot F) - \nabla^2 F \]
\[ \nabla \cdot (\varphi F) = \varphi (\nabla \cdot F) + F \cdot (\nabla \varphi) \]
\[ \nabla \times (\varphi F) = \varphi (\nabla \times F) - F \times (\nabla \varphi) \]
\[ \nabla (F \cdot G) = (F \cdot \nabla) G + (G \cdot \nabla) F + F \times (\nabla \times G) + G \times (\nabla \times F) \]
\[ \nabla \cdot (F \times G) = G \cdot (\nabla \times F) - F \cdot (\nabla \times G) \]
\[ \nabla \times (F \times G) = (G \cdot \nabla) F - (F \cdot \nabla) G + F(\nabla \cdot G) - G(\nabla \cdot F) \]